210 = 14 \times 15 = 5 \times 6 \times 7 = \binom{21}{2} = \binom{10}{4}

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Abstract. It is given all the solutions of the diophantine equations

\[(y - 1)y(y + 1) = \binom{n}{4} \quad \text{and} \quad x(x + 1) = \binom{n}{4}.\]

1. Introduction

The title of this paper illustrates the remarkable fact that the number 210 can be represented simultaneously as a product of two consecutive integers, a product of three consecutive integers, a triangular number, and as a binomial coefficient \(\binom{n}{4}\) in a nontrivial way\(^1\). In other words, 210 is a common solution to the system of diophantine equations

\[(1) \quad x(x + 1) = (y - 1)y(y + 1) = \binom{m}{2} = \binom{n}{4},\]

where we take \(x, y, m, n \in \mathbb{Z}\) without further restrictions, i.e. \(\binom{m}{2} = \frac{1}{2}m(m - 1)\) and \(\binom{n}{4} = \frac{1}{24}n(n - 1)(n - 2)(n - 3)\) are defined for all \(m, n \in \mathbb{Z}\).

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\(^1\)We prefer not to notice that 210 also is the product of the four smallest prime numbers.
The solution 210 occurs for \( x = -15, 14, y = 6, m = -20, 21, n = -7, 10 \). There is one other integer that can be represented in the above mentioned four ways: the number 0 occurs for \( x = -1, 0, y = -1, 0, 1, m = 0, 1, n = 0, 1, 2, 3 \).

In fact, the system (1) consists of six different diophantine equations. We will consider these equations in this paper.

The equation

\[
x(x + 1) = (y - 1)y(y + 1)
\]

has been solved for the first time in 1963 by Mordell [M]. It has only the solutions \((x, y) = (-15, 6), (-3, 2), (-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (2, 2), (14, 6)\).

The equation

\[
x(x + 1) = \binom{m}{2}
\]

is essentially a Pell equation, and hence trivial. Its solutions are given by \((x, m) = (x_i, m_i)\) for \(i = 0, 1, 2, \ldots\), where \(x_{i+1} = 6x_i - x_{i-1} + 2\) and \(m_{i+1} = 6m_i - m_{i-1} - 2\), with four different sets of initial values: \((x_0, m_0, x_1, m_1) = (0, 1, 2, 4), (0, 0, 2, -3), (-1, 1, -3, 4), (-1, 0, -3, -3)\).

The equation

\[
(y - 1)y(y + 1) = \binom{m}{2}
\]

has been solved for the first time in 1989 by Tzanakis and de Weger [TW]. It has only the solutions \((y, m) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -3), (2, 4), (5, -15), (5, 16), (6, -20), (6, 21), (10, -44), (10, 45), (57, -608), (57, 609), (637, -22736), (637, 22737)\).

The equation

\[
\binom{m}{2} = \binom{n}{4}
\]

has been solved independently by the present two authors, [P] and [dW]. The only solutions are \((m, n) = (-20, -7), (-20, 10), (-5, -3), (-5, 6), (-1, -1), (-1, 4), (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, -1), (2, 4), (6, -3), (6, 6), (21, -7), (21, 10)\).

It is the purpose of this note to solve the remaining two equations. We will prove the following two theorems.
Theorem 1. The equation

\[ (y - 1)y(y + 1) = \binom{n}{4} \]

has only the solutions \((y, n) = (-1, 0), (-1, 1), (-1, 2), (-1, 3), (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (6, -7), (6, 10), (22, -21), (22, 24), (26, -24), (26, 27)\).

Theorem 2. The equation

\[ x(x + 1) = \binom{n}{4} \]

has only the solutions \((x, n) = (-15, -7), (-15, 10), (-1, 0), (-1, 1), (-1, 2), (-1, 3), (0, 0), (0, 1), (0, 2), (0, 3), (14, -7), (14, 10)\).

2. Thue equations for Theorem 1

In equation (2) we put \(X = 6y\) and \(Y = \frac{3}{4} \left((2n - 3)^2 - 5\right)\) (notice that \(X, Y \in \mathbb{Z}\)). Then equation (2) is seen to be equivalent to

\[ Y^2 = X^3 - 36X + 9. \]

This equation defines an elliptic curve, that is of rank 2. We are interested in its integral points, but only in those with \(6 \mid X\).

Let \(K = \mathbb{Q}(\theta)\), where \(\theta\) is a root of \(X^3 - 36X + 9\). Then an integral basis of \(K\) is \(\{1, \theta, \frac{1}{3} \theta^2\}\), the class group is \(C_3\), a system of fundamental units is

\[ \epsilon = 1 - 4\theta - 2\frac{1}{3} \theta^2, \quad \eta = 1 - 4\theta + 2\frac{1}{3} \theta^2. \]

The ramifying primes are 3, 11 and 23, and they ramify as follows:

\[ (3) = \mathfrak{p}_3^3, \quad \mathfrak{p}_3 = \left<-12 + \frac{1}{3} \theta^2\right>, \quad (11) = \mathfrak{p}_{11}^2 q_{11}, \quad (23) = \mathfrak{p}_{23}^2 q_{23}, \]

where \(q_{11}, q_{23}\) are non-principal prime ideals. Note that

\[ X^3 - 36X + 9 = (X - \theta) \left(X^2 + \theta X + (\theta^2 - 36)\right), \]

and if a prime ideal \(\mathfrak{p}\) divides both \((X - \theta)\) and \((X^2 + \theta X + (\theta^2 - 36))\), then it divides \((X + 2\theta)(X - \theta) - (X^2 + \theta X + (\theta^2 - 36)) = \langle 3^2 (-4+\right)\).
\[ \frac{1}{3} \theta^2 \) = p_3^8 p_1^2 p_2^3. \]

Since \( 3 \mid X \) and \( \text{ord}_{p_3}(\theta) = 2 \), we have \( \text{ord}_{p_3}(X - \theta) = 2 \), and \( \text{ord}_{p_3}(X^2 + \theta X + (\theta^2 - 36)) = 4 \). Thus from equation (4) we see that there are \( a, b \in \{0, 1\} \) and an integral ideal \( a \) such that
\[ (X - \theta) = p_3^2 p_1^a p_2^b a^2. \]

On taking norms we find \( Y^2 = 3^2 11^a 23^b (N\mathfrak{a})^2 \), so that \( a = b = 0 \). Further it follows that \( a^2 \) is principal, hence so is \( a \). There exist \( m, n \in \{0, 1\} \) such that
\[ X - \theta = \pm \varepsilon^m \eta^n (-12 + \frac{1}{3} \theta^2)^2 \alpha^2, \]
where \( \alpha \) is a generator of \( \mathfrak{a} \).

Now we look at embeddings of \( \mathbb{K} \) into \( \mathbb{R} \). We write \( \theta_1 = -6.12 \ldots, \theta_2 = 0.25 \ldots, \theta_3 = 5.87 \ldots \), and then find that \( \varepsilon_2 \) and \( \varepsilon_3 \) are negative, whereas \( \varepsilon_1 \) and all conjugates of \( \eta \) are positive. Comparing norms, using that \( N(X - \theta) = Y^2 > 0 \) and \( N\alpha = N\mathfrak{a} = 1 \), we see that the \( \pm \)-sign in (5) is \( + \). Further, if \( X \geq 6 \) then \( X - \theta_i > 0 \) for \( i = 1, 2, 3 \), and it follows by studying the signs that \( m = 0 \). Notice that the solutions of (4) with \( X < 6 \) (and \( 6 \mid X \)) are trivially found to be only \( X = -6, 0 \), leading to \( Y = \pm 3 \) in both cases, and further to \((y, n) = (-1, 0), (-1, 1), (-1, 2), (-1, 3), (0, 0), (0, 1), (0, 2), (0, 3)\).

2.1. The case \( n = 0 \)

In (5) we now may put \( \alpha = A + B\theta + C \frac{1}{3} \theta^2 \), and if \( n = 0 \) we then find
\[ X - \theta = \left(-12 + \frac{1}{3} \theta^2\right)^2 \left(A + B\theta + C \frac{1}{3} \theta^2\right)^2. \]

Expanding out and comparing coefficients, we obtain
\[ X = 144A^2 + 72AB + 6AC + 9B^2, \]
\[ 1 = A^2 - 6BC, \]
\[ 0 = 4A^2 + 2AB - C^2. \]

Equation (7) implies that \( A \) is odd, and that \( A \) and \( B \) are coprime. Thus \( A \) and \( 2A + B \) are coprime, and equation (8), written as \( C^2 = 2A(2A + B) \), is seen to imply the existence of \( E, F \in \mathbb{Z} \) with
\[ A = E^2, \quad B = 2F^2 - 2E^2, \quad C = 2EF. \]
Substituting these expressions into (7) we have
\[ E^4 + 24E^3F - 24EF^3 = E(E^3 + 24E^2F - 24F^3) = 1. \]

Clearly \( E = E^3 + 24E^2F - 24F^3 = \pm 1 \), hence this is trivial: the only solutions are given by \( (E, F) = \pm (1, -1), \pm (1, 0), \pm (1, 1) \), leading respectively to \( (A, B, C) = (1, 0, -2), (1, 0, 2), (1, -2, 0) \), and further to \( (X, Y) = (132, \pm 1515), (36, \pm 213), (156, \pm 1947) \), and finally to \( (y, n) = (22, -21), (22, 24), (6, -7), (6, 10), (26, -24), (26, 27) \).

### 2.2. The case \( n = 1 \)

In (5) we again put \( \alpha = A + B\theta + C^1\theta^2 \), and if \( n = 1 \) we then find by \( 1/\eta = 25 - 2\frac{1}{3}\theta^2 \) that
\[ \left(25 - 2\frac{1}{3}\theta^2\right)(X - \theta) = \left(-12 + \frac{1}{3}\theta^2\right)^2 \left(A + B\theta + C^1\theta^2\right)^2. \]

Expanding out and comparing coefficients, we obtain

(9) \[ 25X - 6 = 144A^2 + 72AB + 6AC + 9B^2, \]

(10) \[ 1 = A^2 - 6BC, \]

(11) \[ \frac{2}{3}X = 4A^2 + 2AB - C^2. \]

Now \( 2 \times (9) + 12 \times (10) - 75 \times (11) \) gives

\[ 25C^2 + (4A - 24B)C + (-2AB + 6B^2) = 0. \]

We view this equation as a quadratic equation in \( C \). If it is to have rational solutions, the discriminant must be a square, \( D^2 \) say. Hence
\[ D^2 = (4A - 24B)^2 - 100(-2AB + 6B^2) = 8(A - B)(2A + 3B). \]

If \( p \) is a prime dividing both \( A - B \) and \( 2A + 3B \), then it divides \( 5A \) and \( 5B \), and since \( A \) and \( B \) are coprime, it must be 5. It follows that we can write
\[ A - B = eE^2, \quad 2A + 3B = fF^2 \]
for unknown integers \( E, F \), where for \( (e, f) \) we have four cases:

\[
(e, f) = (1, 2), (2, 1), (5, 10), (10, 5).
\]

So we get

\[
A = \frac{3}{5} eE^2 + \frac{1}{5} fF^2, \quad B = -\frac{2}{5} eE^2 + \frac{1}{5} fF^2,
\]

\[
C = -\frac{6}{25} eE^2 \pm \frac{1}{25} \sqrt{2ef E F} + \frac{2}{25} fF^2, \quad D = 2\sqrt{2ef E F}.
\]

Since \( F \) is defined up to sign, we can replace the \( \pm \) sign by a +. Now we substitute the above expressions into equation (10), and find

\[-27e^2 E^4 + 12e\sqrt{2ef E F} + 90ef E^2 F^2 - 6f\sqrt{2ef E F} - 7f^2 F^4 = 125.
\]

On putting \( U = 5\sqrt{2ef E F}, V = \sqrt{2ef E F} - F \), which are both integers, we get the Thue equation

\[U^4 - 8U^3V - 12U^2V^2 + 136UV^3 - 140V^4 = \frac{2500}{f^2}.
\]

Notice that with \( f = 1, 2, 5, 10 \) we have \( \frac{2500}{f^2} = 2500, 625, 100, 25 \). The following Theorem treats these Thue equations. Its proof is postponed to a forthcoming section.

**Theorem 3. The Thue equations**

\[
f_1(U, V) = U^4 - 8U^3V - 12U^2V^2 + 136UV^3 - 140V^4 = m,
\]

\[
m \in \{25, 100, 625, 2500\}
\]

have only the solutions \((U, V) = \pm(3, 1)\) at \( m = 25 \), and \((U, V) = \pm(5, 0), \pm(5, 2)\) at \( m = 625 \).

The solutions \((U, V) = \pm(3, 1)\) lead to \((e, f) = (5, 10)\), and to non-integral \( E, F \). The solutions \((U, V) = \pm(5, 0)\) lead to \((e, f) = (1, 2)\), \((E, F) = \pm(1, 1)\), \((A, B, C) = (1, 0, 0)\), \((X, Y) = (6, \pm 3)\), and finally to \((y, n) = (1, 0), (1, 1), (1, 2), (1, 3)\). The solutions \((U, V) = \pm(5, 2)\) lead to \((e, f) = (1, 2)\), \((E, F) = \pm(1, -1)\), and then to non-integral \( C \).

This completes the proof of Theorem 1.
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3. Thue equations for Theorem 2

In equation (3) we put $X = 2n - 3$ and $Y = 8x + 4$. Then equation (3) is seen to be equivalent to

\begin{equation}
6Y^2 = X^4 - 10X^2 + 105.
\end{equation}

This equation defines an elliptic curve, that is of rank 2. We are interested in its integral points.

The right hand side of (13) can be written as

\[(X^2 - 5)^2 + 80 = (X^2 - 5 + 4\sqrt{-5})(X^2 - 5 - 4\sqrt{-5}).\]

Let $K = \mathbb{Q}(\sqrt{-5})$. The class group is $C_2$, and we need to know the behaviour of the primes 2, 3 and 5, which is as follows:

\[\langle 2 \rangle = p_2^2, \quad \langle 3 \rangle = p_3 \overline{p}_3, \quad \langle 5 \rangle = p_5^2, \quad \langle \sqrt{-5} \rangle,
\]

where $p_2, p_3$ are non-principal ideals, the bar denotes complex conjugation, and we have the relations

\[\overline{p}_2 = p_2, \quad p_2 p_3 = (1 + \sqrt{-5}), \quad p_3^2 = (2 - \sqrt{-5}).\]

If $p$ is a prime ideal dividing both $\langle X^2 - 5 + \sqrt{-5} \rangle$ and $\langle X^2 - 5 - 4\sqrt{-5} \rangle$, then it divides $\langle (X^2 - 5 + 4\sqrt{-5}) - (X^2 - 5 - 4\sqrt{-5}) \rangle = \langle 8\sqrt{-5} \rangle = p_2^6 p_5$. It follows by (13) that there exist $a, b, c, d \in \{0, 1\}$ and an integral ideal $\alpha$ such that

\[\langle X^2 - 5 + 4\sqrt{-5} \rangle = p_2^a p_3^b p_5^c \alpha^2.
\]

Taking norms we have $6Y^2 = 2^a 3^b 5^c (N\alpha)\alpha^2$, hence $a = 1$, $(b, c) = (1, 0)$ or $(0, 1)$, $d = 0$. Notice that $\text{ord}_{p_2}(X^2 - 1) \geq 6$, and $\text{ord}_{p_2}(-4 + 4\sqrt{-5}) = 5$, so that we find $\text{ord}_{p_2}(\alpha) = 2$. Hence if $\alpha$ is principal we may write $\alpha = (2A + 2B\sqrt{-5})$, and if $\alpha$ is non-principal, then $\alpha/p_2$ is principal, and we may write $\alpha = p_2 \langle A + B\sqrt{-5} \rangle$, where in both cases $A, B \in \mathbb{Z}$. We define $p = 0$ if $\alpha$ is principal, and $p = 1$ if $\alpha$ is non-principal. Then $\alpha^2 = 2^{2-p} \langle A^2 - 5B^2 + 2AB\sqrt{-5} \rangle$. 

3.1. The case \((b, c) = (1, 0)\)

In the case \((b, c) = (1, 0)\), going from ideals to generators, we thus have

\[
\pm 2^p \left( \frac{X^2 - 5}{4} + \sqrt{-5} \right) = (1 + \sqrt{-5}) (A^2 - 5B^2 + 2AB\sqrt{-5}).
\]

Comparing real and imaginary parts we get

\[
(14) \quad \pm 2^p \frac{X^2 - 5}{4} = A^2 - 10AB - 5B^2,
\]

\[
(15) \quad \pm 2^p = A^2 + 2AB - 5B^2.
\]

Then \(4 \times (14) + 5 \times (15)\) yields


Thus the next field to study is \(\mathbb{L} = \mathbb{Q}(\sqrt{70})\). Its class group is \(C_2\), a fundamental unit is \(251 + 30\sqrt{70}\), and the primes \(2, 3, 5\) and \(7\) behave as follows:

\[
\langle 2 \rangle = p_2^2, \quad \langle 3 \rangle = p_3 q_3, \quad \langle 5 \rangle = p_5^2, \quad p_5 = \langle 25 + 3\sqrt{70} \rangle, \quad \langle 7 \rangle = p_7^2,
\]

where \(p_2, p_3, q_3, p_7\) are non-principal prime ideals. If \(p\) is a prime ideal dividing both

\[
\langle 3A - 5B + B\sqrt{70} \rangle \text{ and } \langle 3A - 5B - B\sqrt{70} \rangle,
\]

then it divides

\[
\langle (3A - 5B + B\sqrt{70}) + (3A - 5B - B\sqrt{70}) \rangle = \langle 2(3A - 5B) \rangle \text{ and also}
\]

\[
\langle (3A - 5B + B\sqrt{70}) - (3A - 5B - B\sqrt{70}) \rangle = \langle 2B\sqrt{70} \rangle.
\]

Since \(A\) and \(B\) are relatively prime (by (15)) we find that \(p\) divides \(2, 3, 5\) or \(7\). It follows that there exist \(a, b, c, d, e \in \{0, 1\}\) and an integral ideal \(b\) such that

\[
\langle 3A - 5B + B\sqrt{70} \rangle = p_2^a p_3^b q_3^c p_5^d p_7^e 5^2.
\]

Taking norms we find that \(2^p X^2 = 2^a 3^b 5^c 7^e (N_b)^2\), and thus that \(a = \) \(p = 0\) or \(1\), \(b = c = 0\) or \(1\), \(d = e = 0\). Since \(\langle 3A - 5B + B\sqrt{70} \rangle\), \(p_3 q_3\) and \(b^2\) are principal ideals, it follows that \(a = p = 0\). Then it also follows that in (14) and (15) the ± sign is a +, because \(A^2 + 2AB - 5B^2 = -1\) has no solutions.
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If \( b \) is principal, we may write \( b = (E + F\sqrt{70}) \), and if \( b \) is non-principal, then \( bp_2 \) is principal, and we may write \( bp_2 = (E + F\sqrt{70}) \), where in both cases \( E, F \) are unknown integers. We let \( q = 0 \) if \( b \) is principal, and \( q = 1 \) if \( b \) is non-principal. Then, going from ideals to generators, we can write

\[
\pm 2^q \left( 3A - 5B + B\sqrt{70} \right) = \left( 251 + 30\sqrt{70} \right)^n 3^b \left( E^2 + 70F^2 + 2EF\sqrt{70} \right),
\]

where also \( n \) can be taken to be in \( \{0, 1\} \). As \( A \) and \( B \) are defined up to sign, we may take the \( \pm \) sign to be a +.

3.1.1. The case \( n = 0 \)
In the case \( n = 0 \), writing \( e = 2^{-q}3^b \) (thus \( e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\} \)), and comparing coefficients, we obtain

\[
3A - 5B = e(E^2 + 70F^2),
\]
\[
B = 2eEF,
\]

hence

\[
A = \frac{1}{3} e(E^2 + 10EF + 70F^2).
\]

We substitute these expressions into (15), and thus get

\[
E^4 + 32E^3F + 180E^2F^2 + 2240EF^3 + 4900F^4 = \frac{9}{e^2}.
\]

We prefer to substitute \( E = U - 2V, F = V \), to get somewhat smaller coefficients. Notice that \( U, V \in \mathbb{Z} \). This gives the Thue equations

(16) \quad U^4 + 24U^3V + 12U^2V^2 + 1872UV^3 + 900V^4 = m

for \( m = \frac{9}{e^2} \in \{1, 4, 9, 36\} \). Below we will treat these Thue equations.

3.1.2. The case \( n = 1 \)
In the case \( n = 1 \), again writing \( e = 2^{-q}3^b \) (thus \( e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\} \)), and comparing coefficients, we find

\[
3A - 5B = e(251E^2 + 4200EF + 17570F^2),
\]
\[
B = e(30E^2 + 502EF + 2100F^2),
\]
hence
\[ A = \frac{1}{3} e(401E^2 + 6710EF + 28070F^2). \]

We substitute these expressions into (15), and thus get
\[
192481E^4 + 6441632E^3F + 80841780E^2F^2 + 450914240EF^3 + 943156900F^4 = \frac{9}{e^2}.
\]

We prefer to substitute \( E = 3U - 31V, F = -\frac{5}{14} U + \frac{28}{7} V \), to get much smaller coefficients. Notice that \( U, V \in \mathbb{Z} \). This gives in fact the Thue equations (16), but this time with \( m = \frac{1764}{e^2} \in \{196, 784, 1764, 7056\} \).

In a forthcoming section we will prove the following result.

**Theorem 4.** The Thue equations
\[ f_3(U, V) = U^4 + 24U^3V + 12U^2V^2 + 1872UV^3 + 900V^4 = m, \]
(17)
\[ m \in \{1, 4, 9, 36, 196, 784, 1764, 7056\} \]

have only the solutions \((U, V) = \pm(1, 0)\) at \( m = 1 \).

The solutions \((U, V) = \pm(1, 0)\) lead to \( m = 1, n = 0, e = 3, (E, F) = \pm(1, 0), (A, B) = (1, 0), (X, Y) = (\pm3, \pm4)\), and finally to \((x, n) = (-1, 0), (-1, 3), (0, 0), (0, 3)\).

3.2. The case \((b, c) = (0, 1)\)

In the case \((b, c) = (0, 1)\), going from ideals to generators, we have
\[ \pm 2^p \left( \frac{X^2 - 5}{4} + \sqrt{-5} \right) = (1 - \sqrt{-5}) (A^2 - 5B^2 + 2AB\sqrt{-5}). \]

Comparing real and imaginary parts we get
\[
\pm 2^p \frac{X^2 - 5}{4} = A^2 + 10AB - 5B^2, \tag{18}
\]
\[ \mp 2^p = A^2 - 2AB - 5B^2. \tag{19} \]

Then \(4 \times (18) - 5 \times (19)\) yields
\[ \mp 2^p X^2 = A^2 - 50AB - 5B^2 = (A - 25B)^2 - 630B^2. \]
Again we work in $L = \mathbb{Q}(\sqrt{70})$. If $p$ is a prime ideal dividing both $\langle A - 25B + 3B\sqrt{70} \rangle$ and $\langle A - 25B - 3B\sqrt{70} \rangle$, then as above we see that $p$ divides 2, 3, 5 or 7. It follows that there exist $a, b, c, d, e \in \{0, 1\}$ and an integral ideal $b$ such that

$$\langle A - 25B + 3B\sqrt{70} \rangle = p_2^a p_3^b q_5^c p_7^d p_5^e b^2.$$  

Taking norms we find that $2^p X^2 = 2^a 3^b + 2^5 5^c (N b)^2$, and thus that $a = p = 0$ or 1, $b = c = 0$ or 1, $d = e = 0$. Since $\langle 3A - 5B + B\sqrt{70} \rangle$, $p_3 q_3$ and $b^2$ are principal ideals, it follows that $a = p = 0$. Then it also follows that in (18) and (19) the $\pm$ and $\mp$ signs respectively are $-$ and $+$, because $A^2 - 2AB - 5B^2 = -1$ has no solutions.

If $b$ is principal, we may write $b = \langle E + F\sqrt{70} \rangle$, and if $b$ is non-principal, then $bp_2$ is principal, and we may write $bp_2 = \langle E + F\sqrt{70} \rangle$, where in both cases $E, F$ are unknown integers. We let $q = 0$ if $b$ is principal, and $q = 1$ if $b$ is non-principal. Then, going from ideals to generators, we can write

$$\pm 2^q \left( A - 25B + 3B\sqrt{70} \right) = \left( 251 + 30\sqrt{70} \right) 3^b \left( E^2 + 70F^2 + 2EF\sqrt{70} \right),$$

where also $n$ can be taken to be in $\{0, 1\}$. As $A$ and $B$ are defined up to sign, we may take the $\pm$ sign to be a $+$.

3.2.1. The case $n = 0$

In the case $n = 0$, writing $e = 2^{-q} 3^b$ (thus $e \in \{1, 3, 1/2, 3/2\}$), and comparing coefficients, we obtain

$$A - 25B = e(E^2 + 70F^2), \quad 3B = 2eEF,$$

hence

$$eA = \frac{1}{3} e(3E^2 + 50EF + 210F^2), \quad B = \frac{2}{3} eEF.$$

We substitute these expressions into (19), and thus get

$$E^4 + 32E^3F + \frac{1180}{3} E^2F^2 + 2240EF^3 + 4900F^4 = \frac{1}{e^2}.$$
We prefer to substitute \( E = \frac{1}{3} U - \frac{19}{3} V, F = V \), to get somewhat smaller coefficients. Notice that \( U, V \in \mathbb{Z} \). This gives the Thue equations

\[
U^4 + 20U^3V + 234U^2V^2 + 2492UV^3 - 2423V^4 = m
\]

for \( m = \frac{81}{e^2} \in \{9, 36, 81, 324\} \). Below we will treat these Thue equations.

3.2.2. The case \( n = 1 \)

In the case \( n = 1 \), again writing \( e = 2^{-\gamma}3^b \) (thus \( e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\} \)), and comparing coefficients, we find

\[
A - 25B = e(251E^2 + 4200EF + 17570F^2),
\]

\[
3B = e(30E^2 + 502EF + 2100F^2),
\]

hence

\[
A = \frac{1}{3}e(1503E^2 + 25150EF + 105210F^2),
\]

\[
B = \frac{1}{3}e(30E^2 + 502EF + 2100F^2).
\]

We substitute these expressions into (19), and thus get

\[
240481E^4 + 8048032E^3F + \frac{303005980}{3}E^2F^2 + 563362240EF^3 + 1178356900F^4 = \frac{1}{e^2}.
\]

We prefer to substitute \( E = \frac{5}{3} U - \frac{221}{3} V, F = -\frac{1}{5} U + \frac{44}{5} V \), to get much smaller coefficients. Notice that \( U, V \in \mathbb{Z} \). This gives in fact the Thue equations (20), but this time with \( m = \frac{2025}{e^2} \in \{225, 900, 2025, 8100\} \).

In a forthcoming section we will prove the following result.

**Theorem 5. The Thue equations**

\[
f_3(U, V) = U^4 + 20U^3V + 234U^2V^2 + 2492UV^3 - 2423V^4 = m,
\]

\[
m \in \{9, 36, 81, 225, 324, 900, 2025, 8100\}
\]
have only the solutions \((U, V) = \pm (3, 0)\) at \(m = 81\), and \((U, V) = \pm (1, 1)\) at \(m = 324\), and \((U, V) = \pm (17, -1)\) at \(m = 8100\).

The solutions \((U, V) = \pm (3, 0)\) lead to \(m = 81\), \(e = 1\), \(n = 0\), \((E, F) = \pm (1, 0)\), \((A, B) = (1, 0)\), \((X, Y) = (\pm 1, \pm 4)\), and finally to \((x, n) = (-1, 1), (-1, 2), (0, 1), (0, 2)\). The solutions \((U, V) = \pm (1, 1)\) lead to \(m = 324\), \(e = \frac{1}{2}\), \(n = 0\), \((E, F) = \pm (-6, 1)\), \((A, B) = (3, -2)\), \((X, Y) = (\pm 17, \pm 116)\), and finally to \((x, n) = (-15, -7), (-15, 10), (14, -7), (14, 10)\). The solutions \((U, V) = \pm (17, -1)\) lead to \(m = 8100\), \(e = \frac{1}{2}\), \(n = 1\), and then to non-integral \(F\). This completes the proof of Theorem 2.

4. Solving the Thue equations

In this section we finally prove Theorems 3, 4 and 5, thus completing also the proofs of Theorems 1 and 2. Using the program package KANT (PC-DOS version) we obtain the following results:
<table>
<thead>
<tr>
<th>Equation</th>
<th>Solutions</th>
<th>486PC-CPU-time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(x,y) = 25$</td>
<td>$(-3,-1),(3,1)$</td>
<td>38</td>
</tr>
<tr>
<td>$f_1(x,y) = 100$</td>
<td>-</td>
<td>33</td>
</tr>
<tr>
<td>$f_1(x,y) = 625$</td>
<td>$(-5,-2),(-5,0),(5,0),(5,2)$</td>
<td>71</td>
</tr>
<tr>
<td>$f_1(x,y) = 2500$</td>
<td>-</td>
<td>110</td>
</tr>
<tr>
<td>$f_2(x,y) = 1$</td>
<td>$(-1,0),(1,0)$</td>
<td>15</td>
</tr>
<tr>
<td>$f_2(x,y) = 4$</td>
<td>-</td>
<td>9</td>
</tr>
<tr>
<td>$f_2(x,y) = 9$</td>
<td>-</td>
<td>9</td>
</tr>
<tr>
<td>$f_2(x,y) = 36$</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>$f_2(x,y) = 196$</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>$f_2(x,y) = 784$</td>
<td>-</td>
<td>18</td>
</tr>
<tr>
<td>$f_2(x,y) = 1764$</td>
<td>-</td>
<td>28</td>
</tr>
<tr>
<td>$f_2(x,y) = 7056$</td>
<td>-</td>
<td>23</td>
</tr>
<tr>
<td>$f_3(x,y) = 9$</td>
<td>-</td>
<td>15</td>
</tr>
<tr>
<td>$f_3(x,y) = 36$</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>$f_3(x,y) = 81$</td>
<td>$(-3,0),(3,0)$</td>
<td>23</td>
</tr>
<tr>
<td>$f_3(x,y) = 225$</td>
<td>-</td>
<td>29</td>
</tr>
<tr>
<td>$f_3(x,y) = 324$</td>
<td>$(-1,-1),(1,1)$</td>
<td>45</td>
</tr>
<tr>
<td>$f_3(x,y) = 900$</td>
<td>-</td>
<td>36</td>
</tr>
<tr>
<td>$f_3(x,y) = 2025$</td>
<td>-</td>
<td>60</td>
</tr>
<tr>
<td>$f_3(x,y) = 8100$</td>
<td>$(-17,1),(17,-1)$</td>
<td>198</td>
</tr>
</tbody>
</table>
210 = 14 \times 15 = 5 \times 6 \times 7 = \binom{21}{2} = \binom{19}{1}

References


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